TILTED ALGEBRAS AND SHORT CHAINS OF MODULES

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Dedicated to Idun Reiten on the occasion of her 70th birthday.

Abstract. We provide an affirmative answer for the question raised almost twenty years ago in [22] concerning the characterization of tilted artin algebras by the existence of a sincere finitely generated module which is not the middle of a short chain.

1. Introduction

Let A be an artin algebra over a commutative artin ring R, that is, A is an R-algebra (associative, with identity) which is finitely generated as an R-module. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, and by $K_0(A)$ the Grothendieck group of A. Further, we denote by D the standard duality $\operatorname{Hom}_R(-, E)$ on $\operatorname{mod} A$, where E is a minimal injective cogenerator in $\operatorname{mod} R$. For a module X in $\operatorname{mod} A$ and its minimal projective presentation $P_1 \xrightarrow{f} P_0 \longrightarrow X \longrightarrow 0$ in $\operatorname{mod} A$, the transpose $\operatorname{Tr} X$ of X is the cokernel of the map $\operatorname{Hom}_A(f, A)$ in $\operatorname{mod} A^{\mathrm{op}}$, where A^{op} is the opposite algebra of A. Then we obtain the homological operator $\tau_A = D \operatorname{Tr}$ on modules in $\operatorname{mod} A$, called the Auslander-Reiten translation, playing a fundamental role in the modern representation theory of artin algebras. A module M in $\operatorname{mod} A$ is said to be sincere if every simple right A-module occurs as a composition factor of M. Finally, following [8], [13], A is a tilted algebra if A is an algebra of the form $\operatorname{End}_H(T)$, where H is a hereditary artin algebra and T is a tilting module in $\operatorname{mod} H$, that is, $\operatorname{Ext}^1_H(T, T) = 0$ and the number of pairwise nonisomorphic indecomposable direct summands of T is equal to the rank of $K_0(H)$.

The aim of the article is to establish the following characterization of tilted algebras.

Theorem 1. *An artin algebra A is a tilted algebra if and only if there exists a sincere module M in $\operatorname{mod} A$ such that, for any module X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(X, M) = 0$ or $\operatorname{Hom}_A(M, \tau_A X) = 0$.*

We note that, by [3], [22], a sequence $X \longrightarrow M \longrightarrow \tau_A X$ of nonzero homomorphisms in a module category $\operatorname{mod} A$ with X being indecomposable is called a short chain, and M the middle of this short chain. Therefore, the theorem asserts that an artin algebra A is a tilted algebra if and only if $\operatorname{mod} A$ admits a sincere module

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M which is not the middle of a short chain. Hence, the theorem provides an affirmative answer for the question raised in [22, Section 3]. We would like to mention that some related partial results have been proved by C.M. Ringel [23, p.376] and Ø. Bakke [6].

The tilted algebras play a prominent role in the modern representation theory of algebras and have attracted much attention (see [1], [20], [23], [24], [26], [27] and their cited papers). In particular, the following handy criterion for an algebra to be tilted has been established independently in [19], [28]: an indecomposable artin algebra A is a tilted algebra if and only if the Auslander-Reiten quiver Γ_A of A admits a component $\mathcal{C}$ with a faithful section Δ such that $\mathrm{Hom}_A(X, \tau_A Y) = 0$ for all modules X and Y in Δ. We refer also to [14], [19], [28], [29], [30] for characterizations of distinguished classes of tilted algebras via properties of components of the Auslander-Reiten quivers and to [31] for a homological characterization of tilted algebras. The tilted algebras are also crucial for the classification of finite dimensional selfinjective algebras of finite growth over an algebraically closed field. Namely, the indecomposable algebras in this class are, up to Morita equivalence, socle deformations of the orbit algebras $\widehat{B}/G$ of the repetitive algebras $\widehat{B}$ of tilted algebras B of Dynkin and Euclidean types and admissible infinite cyclic groups G of automorphisms of $\widehat{B}$ (see the survey article [32]). Further, the tilted algebras allow also to recover selfinjective artin algebras whose Auslander-Reiten quiver admits components of a prescribed form. For example, it has been shown in [33] (see also [34], [35]) that, if the Auslander-Reiten quiver of an indecomposable selfinjective artin algebra A admits a generalized standard (in the sense of [29]) acyclic component, then A is Morita equivalent to a socle deformation of an orbit algebra $\widehat{B}/G$ of the repetitive algebra $\widehat{B}$ of a tilted algebra B not of Dynkin type and an admissible infinite cyclic group G of automorphisms of $\widehat{B}$.

The tilted algebras belong to a wider class of algebras formed by the quasitilted algebras, introduced by D. Happel, I. Reiten and S. O. Smalø in [12], which are the endomorphism algebras $\mathrm{End}_{\mathcal{H}}(T)$ of tilting objects T in hereditary abelian categories $\mathcal{H}$. It has been shown in [12, Theorem 2.3] that these are exactly the artin algebras A of global dimension at most 2 and with every module in $\mathrm{ind}\, A$ of projective dimension or injective dimension at most 1. Further, by a result of D. Happel and I. Reiten [11], every quasitilted artin algebra is a tilted algebra or a quasitilted algebra of canonical type. Moreover, by a result of H. Lenzing and A. Skowroński [16], the quasitilted algebras of canonical type are the artin algebras whose Auslander-Reiten quiver admits a separating family of semiregular (ray or coray) tubes. The key step in our proof of the theorem is to show that the module category $\mathrm{mod}\, A$ of a quasitilted but not tilted algebra A does not admit a sincere module which is not the middle of a short chain, applying results from [5], [15], [16], [21], [23], [25].

For background on the representation theory applied here we refer to [1], [4], [23], [26], [27].

2. Preliminaries

In this section we briefly recall some of the notions we will use and present an essential ingredient of the proof of the main theorem of this article. This is concerned with relationship between semiregular tubes and sincere modules which are not the middle of a short chain.

Let A be an artin algebra over a commutative artin ring R. We denote by Γ_A the Auslander-Reiten quiver of A. Recall that Γ_A is a valued translation quiver whose vertices are the isomorphism classes $\{X\}$ of modules X in $\operatorname{ind} A$, the valued arrows of Γ_A describe minimal left almost split morphisms with indecomposable domain and minimal right almost split morphisms with indecomposable codomain, and the translation is given by the Auslander-Reiten translations $\tau_A = D\operatorname{Tr}$ and $\tau_A^- = \operatorname{Tr} D$. We shall not distinguish between a module X in $\operatorname{ind} A$ and the corresponding vertex $\{X\}$ of Γ_A. Following [29], a component $\mathcal{C}$ of Γ_A is said to be generalized standard if $\operatorname{rad}_A^\infty(X,Y) = 0$ for all modules X and Y in $\mathcal{C}$, where $\operatorname{rad}_A^\infty$ is the infinite Jacobson radical of $\operatorname{mod} A$. Moreover, two components $\mathcal{C}$ and $\mathcal{D}$ of Γ_A are said to be orthogonal if $\operatorname{Hom}_A(X,Y) = 0$ and $\operatorname{Hom}_A(Y,X) = 0$ for all modules X in $\mathcal{C}$ and Y in $\mathcal{D}$. A family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be (strongly) separating if the components in Γ_A split into three disjoint families $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$ such that the following conditions are satisfied:

(S1) $\mathcal{C}^A$ is a sincere family of pairwise orthogonal generalized standard components;

(S2) $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\operatorname{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;

(S3) any morphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\operatorname{mod} A$ factors through $\operatorname{add}(\mathcal{C}_i)$ for any $i \in I$.

We then say that $\mathcal{C}^A$ separates $\mathcal{P}^A$ from $\mathcal{Q}^A$ and write

$$\Gamma_A = \mathcal{P}^A \vee \mathcal{C}^A \vee \mathcal{Q}^A.$$

A component $\mathcal{C}$ of Γ_A is said to be preprojective if $\mathcal{C}$ is acyclic (without oriented cycles) and each module in $\mathcal{C}$ belongs to the τ_A-orbit of a projective module. Dually, $\mathcal{C}$ is said to be preinjective if $\mathcal{C}$ is acyclic and each module in $\mathcal{C}$ belongs to the τ_A-orbit of an injective module. Further, $\mathcal{C}$ is called regular if $\mathcal{C}$ contains neither a projective module nor an injective module. Finally, $\mathcal{C}$ is called semiregular if $\mathcal{C}$ does not contain both a projective module and an injective module. By a general result of S. Liu [17] and Y. Zhang [36], a regular component $\mathcal{C}$ contains an oriented cycle if and only if $\mathcal{C}$ is a stable tube, that is, an orbit quiver $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$. Important classes of semiregular components with oriented cycles are formed by the ray tubes, obtained from stable tubes by a finite number (possibly empty) of ray insertions, and the coray tubes obtained from stable tubes by a finite number (possibly empty) of coray insertions (see [23], [27]).

The following characterizations of ray and coray tubes have been established by S. Liu in [18].

Theorem 2. *Let A be an artin algebra and $\mathcal{C}$ be a semiregular component of Γ_A. The following equivalences hold.*

(i) *$\mathcal{C}$ contains an oriented cycle but no injective module if and only if $\mathcal{C}$ is a ray tube.*

(ii) *$\mathcal{C}$ contains an oriented cycle but no projective module if and only if $\mathcal{C}$ is a coray tube.*

The following lemma will play an important role in the proof of our main theorem.

Lemma 3. *Let A be an algebra and M a sincere module in* $\operatorname{mod} A$ *which is not the middle of a short chain. Then the following statements hold.*

(i) $\mathrm{Hom}_A(M, X) = 0$ *for any A-module X in $\mathcal{T}$, where $\mathcal{T}$ is an arbitrary ray tube of Γ_A containing a projective module.*
(ii) $\mathrm{Hom}_A(X, M) = 0$ *for any A-module X in $\mathcal{T}$, where $\mathcal{T}$ is an arbitrary coray tube of Γ_A containing an injective module.*

Proof. We shall prove only (i), because the proof of (ii) is dual. Assume that $\mathrm{Hom}_A(M, X) \neq 0$ for some A-module X from $\mathcal{T}$. Since $\mathcal{T}$ admits at least one projective A-module, we conclude that there exists a projective A-module P in $\mathcal{T}$ lying on an oriented cycle. Let Σ be the sectional path in $\mathcal{T}$ from infinity to P and Ω be the sectional path in $\mathcal{T}$ from X to infinity. Then Σ intersects Ω and there is a common module Y of Σ and Ω different from X and P. Denote by Z the immediate successor of Y on the sectional path from Y to P. Then $\tau_A Z$ is the immediate predecessor of Y on the sectional path from X to Y. Note that all irreducible maps corresponding to the arrows on rays in $\mathcal{T}$ are monomorphisms. Therefore, considering the composition of irreducible monomorphisms corresponding to arrows of Ω forming its subpath from X to $\tau_A Z$ and using our assumption $\mathrm{Hom}_A(M, X) \neq 0$, we get $\mathrm{Hom}_A(M, \tau_A Z) \neq 0$. Further, since M is sincere and not the middle of a short chain, M is faithful by [22, Corollary 3.2]. Hence there is a monomorphism $A_A \to M^r$ for some positive integer r, so a monomorphism $P \to M^r$, because P is a direct summand of A_A. Considering the composition of irreducible homomorphisms corresponding to arrows of Σ forming its subpath from Z to P, we receive that $\mathrm{Hom}_A(Z, P) \neq 0$, by a result of R. Bautista and S. O. Smalø [7]. Hence, $\mathrm{Hom}_A(Z, M^r) \neq 0$, and consequently $\mathrm{Hom}_A(Z, M) \neq 0$. Summing up, we have in $\mathrm{mod}\, A$ a short chain $Z \to M \to \tau_A Z$, a contradiction. This finishes our proof. □

3. Proof of Theorem 1

Let A be an artin algebra over a commutative artin ring R. We may assume (without loss of generality) that A is basic and indecomposable.

Assume first that A is a tilted algebra, that is, $A = \mathrm{End}_H(T)$ for a basic indecomposable hereditary artin algebra H over R and a multiplicity-free tilting module T in $\mathrm{mod}\, H$. Then the tilting H-module T determines the torsion pair $(\mathcal{F}(T), \mathcal{T}(T))$ in $\mathrm{mod}\, H$, with the torsion-free part $\mathcal{F}(T) = \{X \in \mathrm{mod}\, H | \mathrm{Hom}_H(T, X) = 0\}$ and the torsion part $\mathcal{T}(T) = \{X \in \mathrm{mod}\, H | \mathrm{Ext}^1_H(T, X) = 0\}$, and the splitting torsion pair $(\mathcal{Y}(T), \mathcal{X}(T))$ in $\mathrm{mod}\, A$, with the torsion-free part $\mathcal{Y}(T) = \{Y \in \mathrm{mod}\, A | \mathrm{Tor}^A_1(Y, T) = 0\}$ and the torsion part $\mathcal{X}(T) = \{Y \in \mathrm{mod}\, A | Y \otimes_A T = 0\}$. Then, by the Brenner-Butler theorem, the functor $\mathrm{Hom}_H(T, -) : \mathrm{mod}\, H \to \mathrm{mod}\, A$ induces an equivalence of $\mathcal{T}(T)$ with $\mathcal{Y}(T)$, and the functor $\mathrm{Ext}^1_H(T, -) : \mathrm{mod}\, H \to \mathrm{mod}\, A$ induces an equivalence of $\mathcal{F}(T)$ with $\mathcal{X}(T)$ (see [9], [13]). Further, the images $\mathrm{Hom}_H(T, I)$ of the indecomposable injective modules I in $\mathrm{mod}\, H$ via the functor $\mathrm{Hom}_H(T, -)$ belong to one component $\mathcal{C}_T$ of Γ_A, called the connecting component of Γ_A determined by T, and form a faithful section Δ_T of $\mathcal{C}_T$, with Δ_T the opposite valued quiver Q_H^{op} of the valued quiver Q_H of H. Recall that a full connected valued subquiver Σ of a component $\mathcal{C}$ of Γ_A is called a section if Σ has no oriented cycles, is convex in $\mathcal{C}$, and intersects each τ_A-orbit of $\mathcal{C}$ exactly once. Moreover, the section Σ is faithful provided the direct sum of all modules lying on Σ is a faithful A-module. The section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_A has the distinguished property: it connects the torsion-free part $\mathcal{Y}(T)$ with the torsion part $\mathcal{X}(T)$, because every predecessor in $\mathrm{ind}\, A$ of a module $\mathrm{Hom}_H(T, I)$ from Δ_T lies

in $\mathcal{Y}(T)$ and every successor of $\tau_A^- \operatorname{Hom}_H(T, I)$ in $\operatorname{ind} A$ lies in $\mathcal{X}(T)$. Let M_T be the direct sum of all modules lying on Δ_T. Then $M_T = \operatorname{Hom}_H(T, D(H))$ and is a sincere A-module. Suppose M_T is the middle of a short chain $X \to M_T \to \tau_A X$ in $\operatorname{mod} A$. Then X is a predecessor of an indecomposable direct summand M' of M_T in $\operatorname{ind} A$ and consequently $M_T \in \mathcal{Y}(T)$ forces $X \in \mathcal{Y}(T)$. Hence $\tau_A X$ also belongs to $\mathcal{Y}(T)$, since $\mathcal{Y}(T)$ is closed under predecessors in $\operatorname{ind} A$. In particular, $\tau_A X$ does not lie on Δ_T. Then $\operatorname{Hom}_A(M_T, \tau_A X) \neq 0$ implies that there is an indecomposable direct summand M'' of M_T such that $\tau_A X$ is a successor of $\tau_A^{-1} M''$ in $\operatorname{ind} A$. But then $\tau_A^{-1} M'' \in \mathcal{X}(T)$ forces that $\tau_A X \in \mathcal{X}(T)$, because $\mathcal{X}(T)$ is closed under successors in $\operatorname{ind} A$. Hence $\tau_A X$ is simultaneously in $\mathcal{Y}(T)$ and $\mathcal{X}(T)$, a contradiction. Thus M_T is, as wanted, a sincere A-module which is not the middle of a short chain.

Conversely, assume that M is a sincere module in $\operatorname{mod} A$ which is not the middle of a short chain. Then, by [22, Theorem 3.5 and Proposition 3.6], we have that $\operatorname{gl.dim} A \leq 2$ and $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$ for any indecomposable module X in $\operatorname{mod} A$. Thus A is a quasitilted algebra by the characterization given by D. Happel, I. Reiten and S. O. Smalø in [12, Theorem 2.3].

We shall now assume that A is not a tilted algebra and prove that this leads to a contradiction.

Let A be a quasitilted algebra which is not tilted. Then by the result of D. Happel and I. Reiten [11], A is a quasitilted algebra of canonical type. Hence, following [16], the Auslander-Reiten quiver Γ_A of A has a disjoint union decomposition of the form

$$\Gamma_A = \mathcal{P}^A \vee \mathcal{T}^A \vee \mathcal{Q}^A,$$

where $\mathcal{T}^A$ is a separating family of pairwise orthogonal generalized standard semiregular (ray or coray) tubes. Note that the separating family $\mathcal{T}^A$ satisfies the following conditions: $\operatorname{Hom}_A(\mathcal{T}^A, \mathcal{P}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{T}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$ and every homomorphism $f : U \to V$ with U in $\mathcal{P}^A$ and V in $\mathcal{Q}^A$ factorizes through a module W from the additive category $\operatorname{add}(\mathcal{T})$ of any tube $\mathcal{T}$ of $\mathcal{T}^A$.

Let N be an indecomposable direct summand of M. Since N is not the middle of a short chain, N does not lie on a short cycle $X \to N \to X$ in $\operatorname{ind} A$ [10, Lemma 1], and consequently N cannot belong to any stable tube of Γ_A. Thus, in view of Lemma 3, we conclude that M has no direct summands in $\mathcal{T}^A$. Therefore $M = M_P \oplus M_Q$, where M_P is the maximal direct summand of M from the additive category $\operatorname{add}(\mathcal{P}^A)$ of $\mathcal{P}^A$ and M_Q is the maximal direct summand of M from the additive category $\operatorname{add}(\mathcal{Q}^A)$ of $\mathcal{Q}^A$. We claim that either $M_P = 0$ or $M_Q = 0$. Assume that $M_P \neq 0$ and $M_Q \neq 0$. There are three cases to consider (by the structure of the family $\mathcal{T}^A$).

Case 1. $\mathcal{T}^A$ is a separating family of stable tubes. By [15, Theorem 1.1], it follows that A is a concealed canonical algebra. Recall that a concealed canonical algebra is an algebra of the form $B = \operatorname{End}_\Lambda(T)$, where Λ is a canonical algebra in the sense of C. M. Ringel [25] and T is a multiplicity-free tilting module in the additive category $\operatorname{add}(\mathcal{P}^\Lambda)$, for the canonical decomposition $\Gamma_\Lambda = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee \mathcal{Q}^\Lambda$ of Γ_Λ, with $\mathcal{T}^\Lambda$ the canonical infinite separating family of stable tubes of Γ_Λ. Then $\mathcal{P}^A$ is a family of components containing all the indecomposable projective A-modules, $\mathcal{Q}^A$ is a family of components containing all the indecomposable injective A-modules and $\mathcal{T}^A$ separates $\mathcal{P}^A$ from $\mathcal{Q}^A$. Let M' be an indecomposable direct summand of

M_P and M'' be an indecomposable direct summand of M_Q. In this situation we have that $\mathrm{Hom}_A(M', \mathcal{Q}^A) \neq 0$, because for the injective hull $u : M' \to E_A(M')$ of M', $E_A(M')$ is contained in $\mathrm{add}(\mathcal{Q}^A)$. Dually, $\mathrm{Hom}_A(\mathcal{P}^A, M'') \neq 0$, since for the projective cover $h : P_A(M'') \to M''$ of M'', $P_A(M'')$ belongs to $\mathrm{add}(\mathcal{P}^A)$. Thus there are indecomposable A-modules X and Y in an arbitrary given stable tube $\mathcal{T}$ from $\mathcal{T}^A$ such that $\mathrm{Hom}_A(M', X) \neq 0$ and $\mathrm{Hom}_A(Y, M'') \neq 0$.

Let Σ be the infinite sectional path in $\mathcal{T}$ which starts at X and Ω be the infinite sectional path in $\mathcal{T}$ which terminates at Y. Note that since $\mathcal{T}$ is a stable tube, all morphisms corresponding to arrows of Σ are monomorphisms, whereas all morphisms which correspond to arrows of Ω are epimorphisms. Further, Σ intersects Ω and there is a common module V different from X and Y. Denote by Z the immediate successor of V on the path Ω. Then $\tau_A Z$ is the immediate predecessor of V on the path Σ. Thus there are a nonzero monomorphism from X to $\tau_A Z$, which is a composition of irreducible monomorphisms, and a nonzero epimorphism from Z to Y being a composition of irreducible epimorphisms. Therefore, $\mathrm{Hom}_A(M', \tau_A Z) \neq 0$ and $\mathrm{Hom}_A(Z, M'') \neq 0$. In that way we obtain a short chain $Z \to M \to \tau_A Z$ in $\mathrm{mod}\, A$, which contradicts the assumption imposed on M.

Case 2. There exists an indecomposable projective A-module belonging to $\mathcal{T}^A$, but there are no indecomposable injective modules in $\mathcal{T}^A$. Thus A is almost concealed canonical, by [16, Corollary 3.5], and $\mathcal{Q}^A$ contains all indecomposable injective modules. Recall that it means that $A = \mathrm{End}_\Lambda(T)$ for a canonical algebra Λ and a tilting module T from the additive category $\mathrm{add}(\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda)$ of $\mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda$, where $\Gamma_\Lambda = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee \mathcal{Q}^\Lambda$ is the canonical decomposition of Γ_Λ with $\mathcal{T}^\Lambda$ the canonical separating family of stable tubes.

Let M' be an indecomposable direct summand of M_P and $\mathcal{T}$ be a ray tube of the family $\mathcal{T}^A$ which contains a projective module. Again, for an injective hull $M' \to E_A(M')$ of M' in $\mathrm{mod}\, A$, we have $E_A(M') \in \mathrm{add}(\mathcal{Q}^A)$. Using now the separation property of $\mathcal{T}^A$ we conclude that $\mathrm{Hom}_A(M', X) \neq 0$ for a module X in $\mathcal{T}$, which leads to a contradiction with Lemma 3. Therefore, $M_P = 0$. Applying dual arguments we show that, if $\mathcal{T}^A$ contains an indecomposable injective A-module but no indecomposable projective A-modules belong to $\mathcal{T}^A$, then $M_Q = 0$.

Case 3. There exists a ray tube $\mathcal{T}'$ in $\mathcal{T}^A$ containing a projective A-module and a coray tube $\mathcal{T}''$ in $\mathcal{T}^A$ containing an injective A-module. In this case, A is a semiregular branch enlargement of a concealed canonical algebra B with respect to a separating family $\mathcal{T}^B$ of stable tubes in Γ_B (see [16, Theorem 3.4]). Moreover, A admits quotient algebras A_l and A_r such that A_l^{op} is almost concealed canonical with $\mathcal{P}^A = \mathcal{P}^{A_l}$ and A_r is almost concealed canonical with $\mathcal{Q}^A = \mathcal{Q}^{A_r}$. We note that A_l is a branch coextension of B with respect to $\mathcal{T}^B$ and A_r is a branch extension of B with respect to $\mathcal{T}^B$.

We start with the observation that the coray tube $\mathcal{T}''$ contains an injective module I such that $I/\mathrm{soc}\, I$ has an indecomposable direct summand X which is a B-module lying in a stable tube of $\mathcal{T}^B$. Since M is faithful (see again [22, Corollary 3.2]), there is an epimorphism $M^s \to I/\mathrm{soc}\, I$, for some positive integer s, and hence an epimorphism $(M_P)^s \to I/\mathrm{soc}\, I$, because $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{T}^A) = 0$. Clearly, then there is also an epimorphism $(M_P)^s \to X$. Choose a simple B-module S such that there is an epimorphism $X \to S$. Hence there is an epimorphism $M_P \to$

S. Let M' be a direct summand of M_P satisfying the condition $\mathrm{Hom}_A(M', S) \neq 0$. Since S is a B-module, for an injective hull $S \to E_A(S)$ of S in $\mathrm{mod}\, A$, we conclude that $E_A(S)$ is an injective A_r-module, and so $E_A(S) \in \mathcal{Q}^A$. Therefore, we obtain that $\mathrm{Hom}_A(M', E_A(S)) \neq 0$ with $E_A(S)$ in $\mathcal{Q}^A$. Similarly, we may take an indecomposable projective module P in the ray tube $\mathcal{T}'$ such that $\mathrm{rad}\, P$ has a direct summand Y which is a B-module. Let T be a simple submodule of Y. Since there is a monomorphism $A_A \to M^r$, we get a monomorphism $P \to (M_Q)^r$, because $\mathrm{Hom}_A(\mathcal{T}^A, \mathcal{P}^A) = 0$, and consequently there is an embedding $T \to (M_Q)^r$. Then $\mathrm{Hom}_A(P_A(T), M'') \neq 0$ holds for some direct summand M'' of M_Q and the A_l-module $P_A(T) \in \mathcal{P}^A$ such that $P_A(T) \to T$ is a projective cover of T.

Since $\mathcal{T}^A$ separates $\mathcal{P}^A$ from $\mathcal{Q}^A$ in $\mathrm{mod}\, A$, any nonzero homomorphism from M' to $E_A(S)$ factorizes through $\mathcal{T}'$ and any nonzero homomorphism from $P_A(T)$ to M'' has a factorization through $\mathcal{T}''$. But this contradicts Lemma 3.

Summing up, we have shown that:

- if $M_P \neq 0$, then $\mathcal{T}^A$ is a separating family of coray tubes (thus without projectives),
- if $M_Q \neq 0$, then $\mathcal{T}^A$ is a separating family of ray tubes (thus without injectives),
- either $M_P = 0$ or $M_Q = 0$.

By duality, we may assume that $M_P = 0$. Then, $M = M_Q \in \mathrm{add}(\mathcal{Q}^A)$ and $\mathcal{T}^A$ is a separating family of ray tubes. Equivalently, A is an almost concealed canonical algebra.

By general theory of quasitilted algebras of canonical type ([15], [16]), the class of almost concealed canonical algebras divides into three classes: almost concealed canonical algebras of Euclidean type, tubular type and wild type. According to the assumption that A is not a tilted algebra, thus not of Euclidean type, we shall consider only the case of tubular type and of wild type.

(i) Suppose first that A is an almost concealed canonical algebra of tubular type. Then, by [23, Chapter 5] (also [25]), the shape of the Auslander-Reiten quiver Γ_A of A is as follows

$$\Gamma_A = \mathcal{P}(A) \vee \mathcal{T}_0^A \vee \Big(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^A\Big) \vee \mathcal{T}_\infty^A \vee \mathcal{Q}(A),$$

where $\mathcal{P}(A)$ is a preprojective component with a Euclidean section, $\mathcal{Q}(A)$ is a preinjective component with a Euclidean section, $\mathcal{T}_0^A$ is an infinite family of pairwise orthogonal generalized standard ray tubes containing at least one indecomposable projective A-module, $\mathcal{T}_\infty^A$ is an infinite family of pairwise orthogonal generalized standard coray tubes containing at least one indecomposable injective A-module, and $\mathcal{T}_q^A$, for q in the set of positive rational numbers $\mathbb{Q}^+$, is an infinite family of pairwise orthogonal generalized standard faithful stable tubes. Moreover, for $r < s$ in $\mathbb{Q}^+ \cup \{0, \infty\}$, we have $\mathrm{Hom}_A(\mathcal{T}_s^A, \mathcal{T}_r^A) = 0$.

We claim that $M \in \mathrm{add}(\mathcal{Q}(A))$. Let N be an indecomposable direct summand of M. Observe that because N does not lie on a short cycle in $\mathrm{mod}\, A$, N cannot belong to any stable tube from the families $\mathcal{T}_q^A$. Moreover, by Lemma 3, N is neither in a ray tube of $\mathcal{T}_0^A$ with an indecomposable projective A-module nor in a coray tube of $\mathcal{T}_\infty^A$ with an indecomposable injective A-module. Since $M \in \mathrm{add}(\mathcal{Q}^A)$, we

obtain that $N \in \mathcal{Q}(A)$, and hence $M \in \operatorname{add}(\mathcal{Q}(A))$.

Let I be an indecomposable injective A-module from the family $\mathcal{T}_\infty^A$. Since M is a faithful module, there is an epimorphism of the form $M^s \to I$ for some positive integer s. On the other hand, $\operatorname{Hom}_A(\mathcal{Q}(A), \mathcal{T}_\infty^A) = 0$ forces $\operatorname{Hom}_A(M, I) = 0$, a contradiction. This fact excludes the class of almost concealed canonical algebras of tubular type from the class of algebras which admit a sincere finitely generated module which is not the middle of a short chain.

(ii) Let now A be an almost concealed canonical algebra of wild type. Then, by [16] and [21], Γ_A is of the form

$$\Gamma_A = \mathcal{P}^A \vee \mathcal{T}^A \vee \mathcal{Q}^A,$$

where $\mathcal{P}^A$ consists of a unique preprojective component $\mathcal{P}(A)$ and an infinite family of components obtained from components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of ray insertions, $\mathcal{Q}^A$ consists of a unique preinjective component $\mathcal{Q}(A)$ and an infinite family of components obtained from components of the form $\mathbb{Z}\mathbb{A}_\infty$ by a finite number (possibly empty) of coray insertions. By [21], $\mathcal{Q}(A)$ is a preinjective component $\mathcal{Q}(C)$ for some wild concealed algebra C, which is an indecomposable quotient algebra of A. Thus $C = \operatorname{End}_H(T)$ for an indecomposable wild hereditary algebra H and a tilting H-module T from the additive category $\operatorname{add}(\mathcal{P}(H))$ of the preprojective component $\mathcal{P}(H)$ of Γ_H. Moreover, $\mathcal{Q}^A$ contains at least one injective module which does not belong to $\mathcal{Q}(C)$. In fact, there is an indecomposable injective A-module I in $\mathcal{Q}^A \backslash \mathcal{Q}(A)$ such that $I/\operatorname{soc} I$ has an indecomposable direct summand X which is a regular C-module. Denote by $\mathcal{D}$ the component of Γ_A that I belongs to. Since M is a faithful A-module, there is an indecomposable direct summand M_1 of M such that $\operatorname{Hom}_A(M_1, I) \neq 0$. Clearly, M_1 belongs to a component, say $\mathcal{C}$, from $\mathcal{Q}^A \backslash \mathcal{Q}(A)$ ($\mathcal{C}$ may be equal to $\mathcal{D}$).

Applying [21, Theorem 6.4], we obtain that there exists an indecomposable module V in $\mathcal{C}$ such that the left cone $(\to V)$ of V in $\mathcal{C}$ (of all predecessors of V in $\mathcal{C}$) consists entirely of indecomposable regular C-modules and the restriction of τ_A to $(\to V)$ coincides with τ_C. We note also that, by tilting theory, there is an equivalence $\operatorname{add}(\mathcal{R}(H)) \xrightarrow{\sim} \operatorname{add}(\mathcal{R}(C))$ of the additive categories of regular modules over H and over C, induced by the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} C$.

Let Y be a quasi-simple regular C-module from the left cone $(\to V)$ of $\mathcal{C}$. Then, by a result of D. Baer [5] (see also [27, Chapter XVIII, Theorem 2.6]), there exists a positive integer m_0 such that $\operatorname{Hom}_A(X, \tau_A^m Y) = \operatorname{Hom}_C(X, \tau_C^m Y) \neq 0$ for all integers $m \geq m_0$. Consider now the infinite sectional path Σ in $\mathcal{C}$ which terminates at M_1. Then there exists $m \geq m_0$ such that the infinite sectional path Ω in $\mathcal{C}$ which starts at $\tau_A^m Y = \tau_C^m Y$ contains a module $\tau_A Z$ with Z lying on Σ. In particular, applying [7], we get that $\operatorname{Hom}_A(Z, M_1) \neq 0$ and consequently $\operatorname{Hom}_A(Z, M) \neq 0$. We shall show that also $\operatorname{Hom}_A(M, \tau_A Z) \neq 0$. Since $\tau_A Z$ belongs to Ω, there exists in $\operatorname{mod} A$ a short exact sequence

$$0 \to \tau_A^m Y \to \tau_A Z \to W \to 0$$

with $\tau_A W$ on Ω (see [2, Corollary 2.2]), and hence a monomorphism from $\tau_A^m Y$ to $\tau_A Z$. Observe that there are in $\operatorname{mod} A$ epimorphisms $M^s \to I$ and $I \to X$.

Further, since $\mathrm{Hom}_A(X, \tau_A^m Y) \neq 0$, we have $\mathrm{Hom}_A(M^s, \tau_A^m Y) \neq 0$, and hence $\mathrm{Hom}_A(M^s, \tau_A Z) \neq 0$. Therefore, we get $\mathrm{Hom}_A(M, \tau_A Z) \neq 0$. This shows that M is the middle of a short chain in $\mathrm{mod}\, A$. Hence any almost concealed canonical algebra of wild type does not admit a sincere finitely generated module which is not the middle of a short chain.

This finally contradicts the assumption that A is not tilted, which finishes the proof.

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